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An Exploration of Some Special Functions and their Applications

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Abstract: ***Special functions are a class of mathematical functions that fall outside of ele-mentary functions. They have distinct properties and wide-ranging applications in areas such as engineering, applied mathematics, physics, statistics, economics, and finance. In this research paper, I study two popular special functions: the Gamma function Γ(m) and the Beta function β(m, n). I develop some of their important properties and results, prove these properties, and demonstrate some of their practical applications. Understanding the properties and applications of these functions is vital for solving mathematical problems that arise in practical modeling. This study sheds new light on the significance and versatility of Γ(m) and β(m, n), their applications, and mathematical properties.***

Keywords: Special, Functions, Wide-Ranging Applications

I. INTRODUCTION

Special functions play a pivotal role in various branches of mathematics, applied sciences, and engineering. These functions are typically defined by properties that set them apart from elementary functions and possess intricate mathematical properties. They often arise in solving complex problems involving differential equations, integrals, and other mathematical operations. Gamma and Beta functions are two fundamental special functions that find wide arrays of applications across different domains. Miller [7], Askey [3], Krasnigi [6], Sadjang [11][15][16][17], Temme [12], and Pucheta [9]

The Gamma function extends the concept of factorials, while the Beta function generalizes binomial coefficients, with applications in probability theory and statistics. In this paper, I delve into the profound significance and applications of Gamma and Beta functions. The properties of the Gamma function make it a versatile tool for solving problems involving integration, series, and asymptotic analysis. The applications of the Gamma function are widespread. In mathematics, it appears in expressions for volumes of higher-dimensional spheres, solutions to various types of differential equations, and evaluations of complex integrals. In physics, it is used in quantum mechanics, statistical mechanics, and various areas of theoretical physics. Additionally, the Gamma function finds applications in engineering, particularly in fields like signal processing, fluid dynamics, and probability theory.

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The Beta function introduced by Thomas Bayes in the 18*th* century is primarily utilized in probability theory and statistics. It is intimately connected with the concept of the beta distribution, a probability distribution often used to model random variables that are constrained to the interval [0, 1]. In the realm of statistics, the Beta function is a crucial component in the calculation of prior and posterior distributions in Bayesian analysis. Moreover, it finds applications in diverse fields such as economics, biology, and reliability theory. The versatility of the Beta function makes it an indispensable tool in mathematical modeling and statistical analysis, enabling a deeper understanding of various phenomena.

II. DEFINITIONS AND PRELIMINARIES

A. Gamma Function $\Gamma(m)$

The Gamma function is a super powerful version of Factorial function. Inntroduced by Swedish Mathematician Leonhard Euler in 18*th* Century, its applicability spreads into the realm of Real and Complex Numbers.

The Gamma function denoted by $\Gamma(m)$ $\forall\ m \in \mathrm{N}$ is defined by:

$\Gamma(m + 1) = m!$ [1]

The extension of the function for $m \in \mathrm{C}$, where $\mathrm{R}(m) > 0$, is defined by

$$\Gamma(m) = \int_0^\infty x^{m-1}e^{-x}dx \quad \forall\ m > 0 \qquad [2]$$

B. Weierstrass Formula

This formula provides an alternative way to compute the gamma function for complex numbers.

$$\forall\ m \in \mathbb{C},\ \frac{1}{\Gamma(m)} = m \cdot e^{\gamma m}\prod_{p=1}^{\infty}(1+\frac{m}{p}) \cdot e^{-\frac{m}{p}} \qquad [3]$$

where γ is the Euler-Mascheroni constant.

Beta Function $\beta(m, n)$

The classical Beta function, denoted by $\beta(m, n)$ and also called Euler?s integral, is defined by the integral.

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx \qquad [4]$$

$$\forall\ n, m \in \mathbb{C}, where\ \Re(m), \Re(n) > 0$$

Properties of Gamma $\Gamma(m)$ **and Beta** $\beta(m, n)$ **Functions**

0.1 $\forall\ m \in \mathrm{Z} \leq 0$, $\Gamma(m)$ **does not exist**

Proof: By definition,

$$\Gamma(m) = \int_0^\infty x^{m-1}e^{-x}dx = (m-1)!$$

Assume Γ(m) exist for all $m \in Z \leq 0$.

For $m = 0$,

$$\Gamma(0) = \int_0^\infty x^{-1}e^{-x}dx = (-1)!$$

Since (−1)! is undefined, the case that Γ(m) exists for all $m \in Z \leq 0$ is false.

0.2 Γ(m) ∀ m ∈ N *is continuous*

For a function to be continuous the following it most possess the following:

- $f(m)$ must be defined ∀ admissible m. In this case m ∈ N
- $\lim_{m\to a} f(m)$ exists
- $\lim_{m\to a} f(m) = f(a)$

Proof: We will prove this through Induction

Base Case: Show it is true for Γ(1), Γ(2)

$$f(1) = \Gamma(1) = \int_0^\infty x^{1-1}e^{-x}dx = \lim_{b\to\infty} \left|-e^{-x}\right|\Big|_0^b = \lim_{b\to\infty}\left(-e^{-b}\right) - \left(-e^{0}\right) = \lim_{b\to\infty}\left(1-e^{-b}\right)$$

as $b \to \infty, e^{-b} \to 0$

$\therefore \Gamma(1) = \lim_{b\to\infty}\left(1-e^{-b}\right) = 1 - 0 = 1$ which implies, $f(1) = \Gamma(1)$ is defined

$$f(2) = \Gamma(2) = \int_0^\infty xe^{-x}dx = \lim_{x\to\infty}\left(-xe^{-x}\right) + \int_0^\infty e^{-x}dx$$

Following from the solution of Γ(1), as $x \to \infty$, $e{-}x \to 0$
We have

$$f(2) = \Gamma(2) = \lim_{x\to\infty}\left(-xe^{-x}\right) + \int_0^\infty e^{-x}dx = 0 + 1 = 1$$

which implies, $f(2) = \Gamma(2)$ is defined

Inductive Step: Let us assume it is true for $m = k$:

$$f(k) = \Gamma(k) = \int_0^\infty x^{k-1}e^{-x}dx$$

Show it is true for $m = k + 1$:

$$f(k+1) = \Gamma(k+1) = \int_0^\infty x^{k}e^{-x}dx$$

Solving, we get:

$$f(k+1) = \Gamma(k+1) = \lim_{x\to\infty}\left(-x^ke^{-x}\right) + \int_0^\infty kx^{k-1}e^{-x}dx = k\cdot\int_0^\infty x^{k-1}e^{-x}\,dx$$

Recall that $\Gamma(k) = \int_0^\infty x^{k-1}e^{-x}dx$

$$\therefore f(k+1) = k\Gamma(k) \qquad [5]$$

Since $f(k) = \Gamma(k)$ is defined for all k ∈ N, $k\Gamma(k)$ is also defined.

∴ $f(m)$ is defined for all m ∈ N.

Next, we show $\lim_{m\to a} f(m)$ exists:

$$\lim_{m\to a} f(m) = \lim_{m\to a}\Gamma(m) = \lim_{m\to a}\int_0^\infty x^{m-1}e^{-x}dx$$

Let the integrand $xm^{-}1e^{-}x = g_m(x)$.

$$\lim_{m\to a}\int_0^\infty x^{m-1}e^{-x}dx = \lim_{m\to a}\int_0^\infty g_m(x)dx$$

Since $\lim_{m\to a} xm^{-}1e^{-}x = \lim_{m\to a} g_m(x) = xa^{-}1e^{-}x$ exists and is integrable, then following Kamihigashi [5] and the dominated convergence theorem:

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$$\lim_{m\to a} f(m) = \lim_{m\to a} \Gamma(m) = \lim_{m\to a} \int_0^\infty x^{m-1}e^{-x}dx = \int_0^\infty \lim_{m\to a} g_m(x)dx$$
$$= \int_0^\infty \lim_{m\to a} x^{m-1}e^{-x}dx = \int_0^\infty x^{a-1}e^{-x}dx = \Gamma(a)$$

Since $\Gamma(a)$ exists, then $\lim_{m\to a} \int_0^\infty x^{m-1}e^{-x}dx$ must also exist.

Hence, $\lim_{m\to a} f(m)$ exists.

All the conditions to show $\Gamma(m)$ for all $m \in N$ have been met.

0.3 $\Gamma(m)\ \forall\ m \in N$ **is monotone increasing**

Proof:

Assume for $m = k$ where $k \in N f(k) \in N$. Show it is true for $m = k + 1$:

Following from Theorem (II)

$$f(k+1) = k\Gamma(k)$$

Since $f(k) = \Gamma(k)$ is an element of N, $k\Gamma(k)$ is also an element of N.

$\therefore$, $f(m) \in N$ for all $m \in N$.

Next, we prove that $f(m_2) \geq f(m_1)$ where $m_2 \geq m_1$ for all $m_2, m_1 \in N$.

$$f(m_2) \geq f(m_1) = (m_2 - 1)! \geq (m_1 - 1)!$$

Let $m_2 = m_1 + b$, use the fact that

$$(m_2 - 1)! = (m_1 - 1 + b)! = (m_1 - 1)!\binom{m_1 - 1 + b}{b}$$

from the above we clearly see that

$$(m_2 - 1)! = (m_1 - 1 + b)! = (m_1 - 1)!\binom{m_1 - 1 + b}{b} \geq (m_1 - 1)!$$

In summary, we have proved that $\forall\ n \in N$, $f(m) \in N$ and $f(m_2) \geq f(m_1)\ \forall\ m_2, m_1 \in N$.

0.4 for $0 < m < 1$**,**

$$\Gamma(m)\Gamma(1-m) = \frac{\pi}{\sin(\pi m)}$$

Proof: Applying Equation [3]

$$\frac{1}{\Gamma(m)}\frac{1}{\Gamma(-m)} = -m^2e^{\gamma m}e^{-\gamma m}\prod_{p=1}^{\infty}\left[(1+\frac{m}{p})e^{-\frac{m}{p}}(1-\frac{m}{p})e^{\frac{m}{p}}\right]$$

Using the fact that $\Gamma(-m) = \frac{\Gamma(1-m)}{m}$, we have:

$$\frac{1}{\Gamma(m)}\frac{1}{\Gamma(1-m)} = m\prod_{p=1}^{\infty}\left(1 - \frac{m^2}{p^2}\right)$$

From sin $(\pi m) = \pi m\prod_{p=1}^{\infty}\left(1 - \frac{m^2}{p^2}\right)$, on substitution, it gives:

$$\Gamma(m)\Gamma(1-m) = \frac{\pi}{\sin(\pi m)} \quad [6]$$

0.5 $\beta(m, n) = \beta(n, m)$

Proof: By definition,

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$$\beta(n,m)=\int_0^1 t^{n-1}(1-t)^{m-1}\,dt$$

From properties of definite integrals, we have:

$$\int_0^a f(x)\,dx=\int_0^a f(a-x)\,dx.$$

Therefore, we have:

$$\beta(n,m)=\int_0^1 (1-t)^{m-1}(1-(1-t))^{n-1}\,dt=\int_0^1 t^{n-1}(1-t)^{m-1}\,dt=\beta(m,n)dt.$$

Hence, we conclude that $\beta(m, n) = \beta(n, m)$.

0.6 $\beta(n,m)=2\int_0^{\frac{\pi}{2}}\sin^{2n-1}(\theta)\cos^{2m-1}(\theta)\,d\theta$

Proof: From the original equation, we have:

$$\beta(n,m)=\int_0^1 t^{n-1}(1-t)^{m-1}\,dt$$

Substitute t = sin2 θ. Then, dt = 2 sin θ cos θ $d\theta$.

The integral can be rewritten as follows

$$\beta(m,n)=\int_0^{\frac{\pi}{2}}(\sin^2\theta)^{m-1}(\cos^2\theta)^{n-1}\cdot 2\sin\theta\cos\theta\,d\theta$$

Simplifying, we have:

$$\beta(m,n)=2\int_0^{\frac{\pi}{2}}\sin^{2m-1}(\theta)\cos^{2n-1}(\theta)\,d\theta$$

Therefore, we have shown that

$$\beta(m,n)=2\int_0^{\frac{\pi}{2}}\sin^{2m-1}(\theta)\cos^{2n-1}(\theta)\,d\theta \qquad [7]$$

0.7 $\beta(m,n)=\beta(m,n+1)+\beta(m+1,n)$

Proof: By definition, we have:

$$\beta(m,n+1)+\beta(m+1,n)=\int_0^1 t^{m-1}(1-t)^n\,dt+\int_0^1 t^m(1-t)^{n-1}\,dt$$

$$=\int_0^1\left[t^{m-1}(1-t)^n+t^m(1-t)^{n-1}\right]dt$$

$$=\int_0^1 t^{m-1}(1-t)^{n-1}[t+(1-t)]\,dt$$

$$=\int_0^1 t^{m-1}(1-t)^{n-1}\,dt=\beta(m,n)$$

$$\therefore\ \beta(m,n)=\beta(m,n+1)+\beta(m+1,n) \qquad [8]$$

0.8 $\beta(m,n)=\dfrac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$

Proof:

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$$\Gamma(m)\Gamma(n) = \int_0^\infty u^{m-1}e^{-u}du \int_0^\infty v^{n-1}e^{-v}dv = \int_0^\infty \int_0^\infty u^{m-1}v^{n-1}e^{-u-v}du\,dv$$

Change of variables: Let $u = st, v = s(1-t)$

$\therefore$ *du dv* = |*J*|*ds dt*, where |*J*| is the Jacobian symbol.

$$J = \begin{vmatrix} \frac{du}{ds} & \frac{dv}{ds} \\ \frac{dv}{ds} & \frac{dv}{dt} \end{vmatrix} = \begin{vmatrix} t & s \\ 1-t & -s \end{vmatrix} = -s$$

$$\therefore |J| = s$$

$$\Gamma(m)\Gamma(n) = \int_0^\infty \int_0^\infty u^{m-1}v^{n-1}e^{-u-v}du\,dv$$

$$= \int_0^1 \int_0^\infty (st)^{m-1}s^{n-1}(1-t)^{y-1}e^{-st-s+st}s\,ds\,dt = \int_0^1 t^{m-1}(1-t)^{n-1}\,dt \int_0^\infty s^{m-1}s^{n-1+1}e^{-s}\,ds$$

$$= \beta(m,n)\Gamma(m+n)$$

Dividing both sides by $\Gamma(m+n)$, we get the required result

$$\beta(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)} \qquad [9]$$

Applications of Gamma $\Gamma(n)$ and Beta $\beta(n, m)$ Functions

The Gamma and Beta functions have widespread applications across various domains due to their versatile mathematical properties. Some key applications are discussed below:

III. FACTORIALS AND COMBINATION

The Gamma function extends the concept of factorial to non-integer values, enabling calcula-tions related to permutations and combinations. Indeed, $n! = \Gamma(n+1) = n\Gamma(n)$, which can be combined with Euler?s reflection formula and used to compute factorials of negative or positive real numbers. Its extension of the factorial function can be used in combinatorics to calculate combinations and permutations for non-whole values as well, Andrews [2].

Example on factorials: Find $(-\frac{1}{2})!$

$$(-\frac{1}{2})! = \Gamma(-\frac{1}{2}+1) = \Gamma(\frac{1}{2})$$

.Using Equation [6]: with $m = \frac{1}{2}$

$$\Gamma(\frac{1}{2})\,\Gamma(\frac{1}{2}) = \frac{\pi}{\sin(\frac{\pi}{2})} = \pi$$

$$\therefore \Gamma(\frac{1}{2}) = \sqrt{\pi}.\ \text{Hence, } (-\frac{1}{2})! = \sqrt{\pi}.$$

Furthermore,

$$\Gamma(\frac{3}{2}) = \frac{1}{2}\Gamma(\frac{1}{2}) = \frac{1}{2}\sqrt{\pi}$$

$$\Gamma(\frac{5}{2}) = \frac{3}{2}\Gamma(\frac{3}{2}) = \frac{3}{4}\sqrt{\pi}$$

By observation

$$(\frac{1}{2})! = \Gamma(\frac{1}{2}+1) = \Gamma(\frac{3}{2}) = \frac{1}{2}\sqrt{\pi},$$

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which exhibits the special property that $(-\frac{1}{2})! > (\frac{1}{2})!$

Example on combinatorics: Solve $\binom{3.5}{1.5}$

$$\binom{3.5}{1.5} = \frac{3.5!}{(3.5-1.5)!1.5!} = \frac{\Gamma(\frac{9}{2})}{\Gamma(3)\Gamma(\frac{3}{2})} = \frac{105/16\sqrt{\pi}}{2!\frac{1}{2}\sqrt{\pi}} = \frac{105}{16} \approx 6.5716472$$

IV. STATISTICS AND PROBABILITY

In statistical analysis, the Gamma and Beta Functions aid in deriving moments, as demonstrated in the properties of the Gamma and Beta Distributions as in Pishro-Nik [8][14].

A. Gamma Distribution

The Gamma distribution is a mathematical function that helps describe the waiting time or duration of an event. It is commonly used to model situations where we aim to understand the duration of time until an event occurs. For instance, consider waiting for a bus and wanting to estimate how long it might take for the bus to arrive. The Gamma distribution provides insights into the various potential waiting times and their respective probabilities. The Probability Density Function (PDF) for a random variable X following a Gamma distribution is denoted as:

$$f_X(x) = \frac{1}{\Gamma(m)\theta^m} x^{m-1} e^{-\frac{x}{\theta}} \qquad [10]$$

where $x \geq 0$ and we write $X \sim \Gamma(m, \theta)$.

Here, x represents a specific waiting time or duration, such as 5 minutes or 10 minutes. The random variable X represents all the possible waiting times or durations.

The Gamma distribution formula also includes two parameters: m and θ. The parameter m affects the shape of the distribution curve. If m is larger, it means the waiting time tends to be longer on average. When the shape parameter m is equal to 1, we have a special case of the Gamma Distribution called the **Exponential Distribution**. The parameter θ controls the spread of the curve. It determines how spread out the waiting times are.

Examples

$$(1) \int_0^\infty f(x)\,dx = 1$$

Proof: Here $f(x)$ is the PDF $f_X(x)$ of the Gamma Distribution with $m, \theta > 0$ Our integral becomes

$$\int_0^\infty f(x)\,dx = \int_0^\infty \frac{1}{\Gamma(m)\theta^m} x^{m-1} e^{-\frac{x}{\theta}}\,dx$$

By rearranging the terms, we can rewrite the integral as:

$$\int_0^\infty \frac{x^{m-1}e^{-\frac{x}{\theta}}}{\Gamma(m)\theta^m}\,dx$$

Comparing this expression to the definition of the Gamma function, we have:

$$\int_0^\infty \frac{x^{m-1}e^{-\frac{x}{\theta}}}{\Gamma(m)\theta^m}\,dx = \frac{1}{\Gamma(m)\theta^m}\int_0^\infty x^{m-1}e^{-\frac{x}{\theta}}\,dx$$

Let $u = \frac{x}{\theta}$

Substitute into the equation, it becomes

$$\frac{1}{\Gamma(m)\theta^m}\int_0^\infty x^{m-1}e^{-\frac{x}{\theta}}\,dx = \frac{1}{\Gamma(m)\theta^m}\int_0^\infty u \cdot \theta^{m-1}e^{-u}\,du \cdot \theta = \frac{1}{\Gamma(m)\theta^m} \cdot \theta^m \int_0^\infty u^{m-1}e^{-u}$$

Recall that $\int_0^\infty u^{m-1}e^{-u} = \Gamma(m)$ putting into the equation, becomes

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$$\frac{1}{\Gamma(m)} \cdot \Gamma(m) = 1$$

Therefore, we have proven that:

$$\int_0^\infty f(x)\,dx = 1$$

The integral of the PDF *f*(*x*) over the entire range of *x* is equal to 1, which is a fundamental property of probability density functions.

(2) The first moment of *X* written as E(*X*) = *m* · *θ*

Proof.

$$f(x) = \frac{1}{\Gamma(m) \cdot \theta^m} \cdot x^{m-1} \cdot e^{-\frac{x}{\theta}}$$

The mean (E(*X*)) of a random variable *X* can be computed as follows:

$$\mathbb{E}(X) = \int_0^\infty x \cdot f(x)\,dx$$

Substituting the PDF of the gamma distribution, we have:

$$\mathbb{E}(X) = \int_0^\infty x \cdot \left(\frac{1}{\Gamma(m) \cdot \theta^k} \cdot x^{m-1} \cdot e^{-\frac{x}{\theta}} \right) dx$$

Simplifying the expression, we obtain:

$$\mathbb{E}(X) = \frac{1}{\Gamma(m) \cdot \theta^m} \int_0^\infty x^m \cdot e^{-\frac{x}{\theta}}\,dx$$

Let $u = \frac{x}{\theta}$

From "Example 1", we have that

$$\mathbb{E}(X) = \frac{1}{\Gamma(m) \cdot \theta^m} \int_0^\infty x^m \cdot e^{-\frac{x}{\theta}}\,dx = \frac{1}{\Gamma(m) \cdot \theta^m} \int_0^\infty u^m \theta^m e^{-u}\,du \cdot \theta$$

Simplifying further,

$$\mathbb{E}(X) = \frac{1}{\Gamma(m) \cdot \theta^m} \cdot \theta^{m+1} \int_0^\infty u^m e^{-u}\,du = \frac{\theta}{\Gamma(m)} \int_0^\infty u^m e^{-u}\,du$$

Recall that $\int_0^\infty u^m e^{-u}\,du = \Gamma(m+1) \text{ and } \Gamma(m+1) = m \cdot \Gamma(m)$

$$\frac{1}{\Gamma(m)} \cdot \Gamma(m) = 1$$

The equation becomes

$$\frac{\theta}{\Gamma(m)} \cdot \Gamma(m+1) = \frac{\theta}{\Gamma(m)} \cdot m \cdot \Gamma(m) = m \cdot \theta$$

$$\therefore\ \mathbb{E}(X) = m \cdot \theta$$ [11]

(3) The second moment of *X* written as *V ar*(*X*) = *m* · *θ*2

Proof: The variance of a random variable *X* is given by:

$$Var(X) = \mathbb{E}(X^2) - [\mathbb{E}(X)]^2$$

Compute the expected value of $$\mathbb{E}(X^2) = \int_0^\infty x^2 \cdot f(x)\,dx$$

$$= \frac{1}{\Gamma(m)\cdot\theta^m}\int_0^\infty x^2 \cdot x^{m-1}\cdot e^{-\frac{x}{\theta}}\,dx = \frac{1}{\Gamma(m)\cdot\theta^m}\int_0^\infty x^{m+1}\cdot e^{-\frac{x}{\theta}}\,dx$$

Once again, substitute $u = \frac{x}{\theta}$ into the integral

$$\frac{1}{\Gamma(m)\cdot\theta^m}\int_0^\infty x^{m+1}\cdot e^{-\frac{x}{\theta}}\,dx = \frac{1}{\Gamma(m)\cdot\theta^m}\int_0^\infty u^{m+1}\theta^{m+2}e^{-u}\,du$$

$$= \frac{1}{\Gamma(m)\cdot\theta^m}\cdot\theta^{m+2}\int_0^\infty u^{m+1}e^{-u}\,du = \frac{\theta^2}{\Gamma(m)}\cdot\Gamma(m+2) = (m)\cdot(m+1)\cdot\theta^2$$

From equation [11], we know that E(X) = $m\cdot\theta$, ∴ [E(X)]2 = ($m\cdot\theta$)2

$$Var(X) = \mathbb{E}(X^2) - [\mathbb{E}(X)]^2 = (m)\cdot(m+1)\cdot\theta^2 - (m\cdot\theta)^2 = m\cdot\theta^2$$

$$\therefore Var(X) = \mathbb{E}(X^2) = m\cdot\theta^2 \quad [12]$$

V. BETA DISTRIBUTION

Now, let's consider a random variable called X, which represents the proportion of red marbles we randomly pick from a bag. It's a way of measuring how many red marbles we get compared to the total number of marbles we pick.

The Beta distribution can help us understand the different probabilities associated with the random variable X. It tells us how likely it is to get various proportions of red marbles when we randomly select marbles from the bag.

The formula for the Beta distribution is:

$$f(x) = \frac{1}{\beta(m,n)}x^{m-1}(1-x)^n$$

In this formula, x represents the proportion of red marbles we're interested in, akin to the fraction of red marbles among all the marbles we pick. It's similar to asking, 'What's the chance of selecting a specific proportion of red marbles?' It represents the different values that X can take on.

The parameters m and n in the formula help shape the distribution curve and determine how the probabilities are spread out. They control the behavior of the Beta distribution and enable us to adjust the probabilities according to our needs.

The function $f(x)$ is a mathematical expression that indicates the probability of a specific proportion x occurring. It's like a *special* machine that takes in a proportion x as input and yields the probability as output. By inputting different values for x, we can calculate the probability of each proportion occurring.

Therefore, the Beta distribution assists us in comprehending the probabilities or likelihoods of various proportions occurring, such as *the* proportion of red marbles in a bag. It serves as a tool enabling us to analyze and predict the likelihood of diverse outcomes.

Examples

$$(1) \quad \int_0^1 f(x)\,dx = 1$$

Proof

Substitute the PDF of the beta distribution into the integral,

$$\int_0^1 \frac{1}{\beta(m,n)}\cdot x^{m-1}\cdot(1-x)^{n-1}\,dx$$

Let's express the beta function in terms of the gamma function:

$$\int_0^1 \frac{1}{\frac{\Gamma(m)\cdot\Gamma(n)}{\Gamma(m+n)}}\cdot x^{m-1}\cdot(1-x)^{n-1}\,dx = \frac{\Gamma(m+n)}{\Gamma(m)\cdot\Gamma(n)}\cdot\int_0^1 x^{m-1}\cdot(1-x)^{n-1}\,dx$$

We can rewrite the integral as a beta function:

$$\frac{\Gamma(m+n)}{\Gamma(m)\cdot\Gamma(n)}\cdot\int_0^1 x^{m-1}\cdot(1-x)^{n-1}=\frac{\Gamma(m+n)}{\Gamma(m)\cdot\Gamma(n)}\cdot\beta(m,n)=\frac{\Gamma(m+n)}{\Gamma(m)\cdot\Gamma(n)}\cdot\frac{\Gamma(m)\cdot\Gamma(n)}{\Gamma(m+n)}=1$$

$$\therefore \int_0^1 f(x)\,dx=1$$

Hence, we have proved that the integral of the PDF of a beta distribution over its support interval [0, 1] is equal to 1.

The first moment of X written as $$\mathbb{E}(X)=\frac{m}{m+n}$$

Proof

$$\mathbb{E}(X)=\int_0^1 x\cdot f(x)\,dx$$

Substituting the PDF into the mean formula, we have:

$$\mathbb{E}(X)=\int_0^1 x\cdot\frac{1}{\beta(m,n)}\cdot x^{m-1}\cdot(1-x)^{n-1}\,dx$$

Express in terms of the Gamma Function

$$\mathbb{E}(X)=\frac{\Gamma(m+n)}{\Gamma(m)\cdot\Gamma(n)}\cdot\int_0^1 x^m\cdot(1-x)^{n-1}\,dx=\frac{\Gamma(m+n)}{\Gamma(m)\cdot\Gamma(n)}\cdot\beta(m+1,n)=\frac{\Gamma(m+n)}{\Gamma(m)\cdot\Gamma(n)}\cdot\frac{\Gamma(m+1)\cdot\Gamma(n)}{\Gamma(m+n+1)}$$

$$=\frac{\Gamma(m+n)}{\Gamma(m)\cdot\Gamma(n)}\cdot\frac{m\cdot\Gamma(m)\cdot\Gamma(n)}{(m+n)\cdot\Gamma(m+n)}$$

Canceling out the common terms, we obtain:

$$\mathbb{E}(X)=\frac{m}{m+n} \qquad [13]$$

Canceling out the common terms, we obtain:

(3) The Second moment of X written as $$Var(X)=\frac{mn}{(m+n)^2(m+n+1)}$$

Proof

Recall that variance of a random variable X is given by:

$$Var(X)=\mathbb{E}(X^2)-[\mathbb{E}(X)]^2$$

Compute the expected value of $$\mathbb{E}(X^2)=\int_0^1 x^2\cdot f(x)\,dx$$

$$=\int_0^1 x^2\cdot\frac{1}{\beta(m,n)}\cdot x^{m-1}\cdot(1-x)^{n-1}\,dx=\int_0^1 x^2\cdot\frac{1}{\frac{\Gamma(m)\cdot\Gamma(n)}{\Gamma(m+n)}}\cdot x^{m+1}\cdot(1-x)^{n-1}\,x$$

$$=\frac{\Gamma(m+n)}{\Gamma(m)\cdot\Gamma(n)}\cdot\int_0^1 x^{m+1}\cdot(1-x)^{n-1}\,dx=\frac{\Gamma(m+n)}{\Gamma(m)\cdot\Gamma(n)}\cdot\beta(m+2,n)=\frac{\Gamma(m+n)}{\Gamma(m)\cdot\Gamma(n)}\cdot\frac{\Gamma(m+2)\cdot\Gamma(n)}{\Gamma(m+n+2)}$$

$$=\frac{\Gamma(m+n)}{\Gamma(m)\cdot\Gamma(n)}\cdot\frac{(m+1)(m)\cdot\Gamma(m)\cdot\Gamma(n)}{(m+n+1)(m+n)\Gamma(m+n)}$$

Canceling out the common terms, we have

$$\mathbb{E}(X^2)=\frac{(m+1)(m)}{(m+n+1)(m+n)}$$

From Equation [13],

$$\mathbb{E}(X)=\frac{m}{m+n}\quad\therefore[\mathbb{E}(X)]^2=\left(\frac{m}{m+n}\right)^2$$

Following from $$Var(X)=\mathbb{E}(X^2)-[\mathbb{E}(X)]^2$$

$$Var(X) = \frac{(m+1)(m)}{(m+n+1)(m+n)} - (\frac{m}{m+n})^2 = \frac{(m^2+m)(m+n) - (m^2)(m+n+1)}{(m+n)^2(m+n+1)}$$

$$= \frac{(m^3+m^2n+m^2+mn) - (m^3+m^2n+m^2)}{(m+n)^2(m+n+1)}$$

Simplifying further, we have

$$Var(X) = \frac{mn}{(m+n)^2(m+n+1)} \qquad [14]$$

The Standard Uniform Distribution

The standard uniform distribution is a special version of the beta distribution where all outcomes within a certain range have an equal chance of occurring. Given that the Beta function, $\beta(m, n)$, is defined as:

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$$

Let $f(x)$ be the Beta distribution for $m, n = 1$

$$f(x) = \beta(1,1) = \int_0^1 x^{1-1}(1-x)^{1-1}\,dx = \int_0^1 x^0 \cdot 1^0\,dx = \int_0^1 1\,dx = 1$$

$$\therefore\ f(x) = 1$$

This means that for any value of x within the range of the standard uniform distribution (usually 0 to 1), the probability density *function* is always equal to 1, which reflects the fact that all outcomes within the range have an equal chance of occurring.

VI. ENGINEERING

The Gamma and Beta functions have numerous applications in engineering disciplines. Within fields like signal processing and fluid mechanics, they aid in modeling real-world systems and solving differential equations. They are also applied *in* reliability engineering, which deals with the study of systems and their ability to perform their intended functions over a specified period. The Beta distribution is commonly used to model the reliability of systems and to calculate probabilities related to their performance.

Example

The *reliability* function $R(t)$ is defined as the probability that the system survives beyond time t. Mathematically, it can be expressed as:

$$R(t) = P(X > t) = \int_t^\infty f(x)dx$$

where X represents the lifetime of the system.

To calculate the reliability function for the Beta distribution, we integrate the PDF from t to infinity. However, since the Beta *distribution* is defined on the interval [0, 1], we consider the complement of the cumulative distribution function (CDF) from 0 to t. Thus, the reliability function can be expressed as:

$$R(t) = 1 - F(t) = 1 - \int_0^t f(x)dx$$

Substituting the PDF of the Beta distribution, we have:

$$R(t) = 1 - \int_0^t \frac{x^{m-1}(1-x)^{n-1}}{\beta(m,n)}dx$$

Further evaluation depends on the values of m and n.

The Gamma and Beta functions find applications beyond combinatorics, engineering, and statistics. They are vital in physics, *economics*, econometrics, image processing, machine learning, and more, and lead to other classes and derived special functions that smoothly extend gamma, beta and other special functions, see Chand [4], Abubakar [1][13], and Rahman [10]. With their versatility, these functions play a crucial role in diverse fields, making them indispensable mathematical tools with vast applications across various scientific and computational disciplines.

VII. CONCLUSION

In this research, we delved into the profound mathematical universe of special functions, with a focus on the Gamma and Beta functions. Our exploration revealed their intricate properties and significant applications across various fields such as engineering, statistics, and physics. We uncovered that the Gamma function and Beta function are not just extensions *of* the factorial concept to complex numbers, but a cornerstone in solving complex integrals and differential equations. Their special properties such as continuity, monotonicity, and their relationship with trigonometric identities, demonstrate their versatility and indispensability in mathematical modeling and problem-solving, making their presence felt in the realms of quantum mechanics and signal processing, probability theory and statistics, underscoring its importance. The potential for further research is vast.

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The applications of the Gamma and Beta functions can be extended to new areas beyond combinatorics, engineering, and statistics such as machine learning and econometrics, providing fresh perspectives and innovative solutions to contemporary challenges. With their versatility, these functions play a crucial role in diverse fields, making them indispensable mathematical tools with vast applications across various scientific and computational disciplines.

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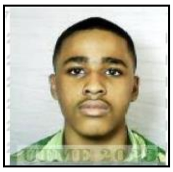
AUTHOR PROFILE



Mfonobong Peter is an early career researcher, instructor, and problem-solver who, in the last 3 years, has competed in several academic contests, including national and international academic Olympiads. In 2021, he won Gold medal in the Kangourou Mathematics Competition and a Certificate of Distinction in the Canadian Senior Mathematics Contest. He secured a Silver Medal consecutively in 2021 and 2022 in the International Youth Mathematics Challenge and a Certificate of Distinction in the Euclid Contest and a School Medal in Euclid in 2023, among several other honors in mathematics between 2021 to 2023. From 2017 to 2021, he participated in the American Mathematics Competitions and excelled, including winning school level medals: 1 Gold Medal in AMC 8 in 2017 and 4 Bronze Medals in AMC 8, 10 and 12 over the period from 2018 to 2021. As a young man of many interests, he is also active in the fields of physics and economics. In 2021 and 2023, as student lead, he led Nigeria to the International Economics Olympiad and represented the country in this contest, having previously won a Certificate of Distinction and Higher Distinction in the World Economics Cup in 2021 and 2023, Bronze Medal in World Economics Cup in 2022, and Gold Medal in the National Economics Olympiad Third Round in 2021 and 2022. In physics, he has made contributions to teaching the subject. He established an intensive academic foundation dedicated to teaching physics and other STEM subjects. Many teenagers in his community have benefitted from his exposition and ability to breakdown difficult concepts into pieces for maximum comprehension for students at all levels of academic abilities and this stands as a testament to his aptitude and dedication to mastering complex mathematical concepts. Beyond his academic pursuits, Mfonobong's commitment to community service reflects his compassionate nature and desire to uplift others. He epitomizes maturity through his academic achievements and his dedication to creating positive change. In recognition of his interest in scientific research and mathematics, Mfonobong was recently invited to join the Artificial Intelligence and Robotics Laboratory of Professor Yinka-Banjo, a professor of AI, Machine Learning and Robotics at the University of Lagos. He was also admitted to the special academic training at BAUM TENPERS Institute Virginia where he completed his research on “An Exploration of Some Special Functions and Their Applications” and is currently working on other research topics of interest and taking relevant prerequisite coursework to help advance his research.